Summary of Differentials, etc.

To work with differentiation and differentials you do not need an extensive knowledge of the theory behind them. You may use the differential operation and its inverse (integration) as another algebraic operation. You can use the differential notations to rearrange equations in order to make it possible to use the inverse operation. Therefore, for practical reasons, you may assume in the following that these descriptions of the differentiation and its inverse are the definitions of a valid algebraic operation. (Remember your practice with the functions flip and funky?)

Where applicable, the differential operation is defined for a power of x by:

$$x = at^n$$

applying the differentiation operation to both sides:

$$\frac{dx}{dt} = nat^{(n-1)}$$

an alternative designation splits the operation $\frac{d}{dt}$ into two parts for rearrangement. These are then referred to as differentials. This symbolism will yield the following:

$$dx = nat^{(n-1)}dt$$ (Note that d is used as a symbol.)

Therefore:

if: $x = at$

then: $\frac{dx}{dt} = a$ and $dx = adt$

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if: $x = at^2$

then: $\frac{dx}{dt} = 2at$ and $dx = 2atdt$

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if: $x = at^3$

then: $\frac{dx}{dt} = 3at^2$ and $dx = 3at^2dt$

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if: $x = a$ $(= at^0)$

then: $\frac{dx}{dt} = 0$

if: $x = at^{-1}$ $\left(= \frac{a}{t} \right)$

then: $\frac{dx}{dt} = -at^{-2}$ $\left(= \frac{a}{t^2} \right)$

if: $x = at^{0.001}$

then: $\frac{dx}{dt} = 0.001at^{-.999}$

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if: $x = at^{\pm 0.00001}$

then: $\frac{dx}{dt} = \pm 0.00001at^{\pm 0.00001-1}$

Notice in differentiating one can obtain any power of t in the answer by selecting n properly **except for t^{-1} exactly!**

The operation of differentiation is distributive across + and -:

$$\frac{d(x + y)}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

thus if: $x = at + bt^2$

then: $\frac{dx}{dt} = a + 2bt$

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and if: $x = at + bt^2 + c + dt^{-1}$

then: $\frac{dx}{dt} = a + 2bt - dt^{-2}$ (what happened to c?)

The inverse operation can be seen from the above examples:

if: $\frac{dx}{dt} = 0$

then: $x = c$ $(= at^0)$ where c is a constant

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if: $\frac{dx}{dt} = 2at$ or $dx = 2atdt$

then: $x = at^2 + c$

What! Where did the c come from? One way of viewing this is that the original equation was:

$$\frac{dx}{dt} = 2at + 0$$

and therefore one must determine by other means what c is.

In General so long as $m \neq -1$ exactly

If: $\frac{dx}{dt} = at^m$ or $dx = at^m dt$

Then: $x = \frac{at^{m+1}}{m} + c$

This inverse operation is also distributive across + and -.

if: $\frac{dx}{dt} = a + 2bt - dt^{-2}$ or $dx = (a + 2bt - dt^{-2})dt$

Then: $x = at + bt^2 + c + dt^{-1}$ (Oh! Hello c)

What to do about m = -1 exactly.

There is a function call the natural logarithm, *ln*, which will generate the results we seek. (To derive all of this would take too much work for us here, we just want a working knowledge of the functions *ln* and e.) That is:

if: $x = a\,ln\,t$

then: $\frac{dx}{dt} = at^{-1}$ ($= \frac{a}{t}$)

and therefore:

if: $\frac{dx}{dt} = \frac{a}{t}$

then: $x = a\,ln\,t + c$

(or: $x + C = a\,ln\,t$)

Alternatively, one can write:

$$dx = \frac{adt}{t} = a\,d(ln\,t) \text{ to give } x + C = a\,ln\,t$$

The inverse function of *ln* is the e function:

$$ln(e^x) = x \quad \text{and} \quad e^{ln\,x} = x$$

From this information, you could derive (by using substitutions) the following:

$$\frac{de^{at}}{dt} = ae^{at}$$

(Hint: use the substitution $y = e^{at}$ and therefore $t = \frac{ln\,y}{a}$)

Strategy in solving problems

As with all algebraic problem solving, one usually isolates the individual unknowns on opposite sides of the equation before using the differential operator or its inverse. Thus put all x's on the side of the equation that has dx in the numerator and all t's on the side of the equation that has dt in the numerator. (By the way, don't get stuck in your thinking of associating x or t with these operations. Remember x and t here are simply arbitrarily chosen letters.)

Example: $xt\frac{dx}{dt} = t^2$

isolate the dx and dt in the numerators:

$$xt\,dx = t^2\,dt$$

place the x's with the dx and the t's with the dt:

$$x\,dx = t\,dt$$

now use the inverse function:

$$\tfrac{1}{2}x^2 = \tfrac{1}{2}t^2 + c$$

Example 2:

$$t^3\frac{dx}{dt} = xt^2$$

$$t^3\,dx = xt^2dt$$

$$\frac{dx}{x} = \frac{dt}{t}$$

$$ln\,x = ln\,t + c$$

Example 3:

$$\frac{dx}{dt} = -kx$$

$$\frac{dx}{x} = -kdt$$

$$lnx + c = -kt$$

or: $\frac{x}{x_o} = e^{-k(t-to)}$ **OOPS! Where did this come from?**